3 ENERGY CALCULATION

3.1 Introduction

Energy changes determined in *UDEC* are performed for the intact rock, the joints and for the work done on boundaries. The energy terms calculated here use the same general nomenclature as those used by Salamon (1984).

Since *UDEC* uses an incremental solution procedure, the equations of motion are solved at each mass point in the body at every timestep. The incremental change in energy components is determined at each timestep as the system attempts to come to equilibrium. *UDEC* also keeps a running sum of each component.

The *UDEC* model itself has finite boundaries that must be taken into account in the energy analysis. The energy analysis requires that the outer boundary of the model be free to deform so that the work done at the boundary can be determined. This requires that either a stress (force) or boundary-element boundary be used.

3.2 Energy Balance

The total energy balance can be expressed in terms of the released energy (W_r), which is the difference between the work done at the boundary of the model and the total stored and dissipated strain energies:

$$W_r = W - (U_c + U_b + W_j + W_p) \tag{3.1}$$

where W_r = released energy;

W = total boundary loading work supplied to the system;

U_c = total stored strain energy in material;

U_b = total change in potential energy of the system;

W_j = total dissipated energy in joint shear; and

W_p = total dissipated work in plastic deformation of intact rock.

A second calculation of released energy can be made based on the kinetic energy, mass damping work, the work performed at viscous boundaries, and the strain energy in excavated material:

$$W_r = U_k + W_k + W_v + U_m \tag{3.2}$$

where U_k = current value of kinetic energy in the system;

W_k = total work dissipated by mass damping;

W_v = work done by viscous (nonreflecting) boundaries; and

U_m = total strain energy in excavated material.

This second form of the released energy is particularly useful for dynamic problems since the released kinetic energy is easily calculated.

The definitions of these individual terms, and the method of their calculation, are given in the following section.

3.3 Calculation of Individual Energy Components

3.3.1 Total Boundary Loading Work (W)

The work done on the external boundaries of the *UDEC* block structure is calculated from the boundary gridpoint forces and displacements. Either a stress (force) or boundary-element outer boundary must be used. For dynamic problems, the stress boundary is replaced by nonreflecting (viscous) boundaries after initial stress equilibrium is achieved. The work done at a gridpoint is calculated:.

$$W_{gi} = F_{xi}\, u_{xi} + F_{yi}\, u_{yi} \tag{3.3}$$

where W_{gi} = work done at gridpoint $\boldsymbol{i}$ on the outer boundary;

F_{xi} = x-oriented force at the gridpoint;

u_{xi} = x-oriented displacement at the gridpoint;

F_{yi} = y-oriented force at the gridpoint; and

u_{yi} = y-oriented displacement at the gridpoint.

The total boundary work, W_l, done during timestep l is the sum of the work done for all gridpoints on the boundary:

$$W_l = \sum_{i=1}^{ngp} W_{gi} \tag{3.4}$$

where ngp = the number of gridpoints on the boundary.

The boundary work is summed for all timesteps:

$$W = \sum_{l=1}^{nt} W_l \tag{3.5}$$

where nt = the number of timesteps. W will approach a constant value as the system approaches equilibrium.

The rate of boundary loading work, ΔW, is simply the boundary work done per timestep:

$$\Delta W = \frac{W_{l+1} - W_l}{(\Delta t)} \tag{3.6}$$

where Δt is the timestep.

Histories of the total work, W, and incremental work, ΔW, are kept during a simulation.

3.3.2 Potential Energy (U_b)

The change in gravitational potential energy is calculated from the gridpoint gravitational forces and the displacements of the gridpoints. The total potential energy is summed for incremental displacements of all gridpoints:

$$U_{gi} = m_i \, [g_x \, u_{xi} + g_y \, u_{yi}] \tag{3.7}$$

where U_{gi} = the potential energy of gridpoint ***i***;

m_i = the mass of gridpoint ***i***;

u_{xi}, u_{yi} = displacement components of gridpoint ***i***; and

g_x, g_y = accelerations in the x- and y-directions (usually gravity).

The total potential energy is found by summing the energy for all gridpoints, ***i***, at a given timestep, k:

$$U_b = \sum_{i=1}^{ngp} U_{gi} \tag{3.8}$$

The total U_b is kept for all timesteps:

$$U_b = \sum_{j=1}^{nt} U_{bj} \tag{3.9}$$

where nt is the number of timesteps; U_b will approach a constant value as the model approaches equilibrium.

3.3.3 Kinetic Energy (U_k)

The kinetic energy is determined for each gridpoint at each timestep, and is summed for all gridpoints at that timestep. A running total of the kinetic energy is not kept; so, as the system approaches equilibrium, the kinetic energy will approach zero. The kinetic energy is given by

$$U_k = \sum_{i=1}^{ngp} \frac{1}{2} m_i \, (\dot{u}_i)^2 \tag{3.10}$$

where U_k = kinetic energy of all gridpoints in a given timestep;

m_i = mass of gridpoint ***i***; and

$\dot{u}_i$ = velocity at gridpoint ***i***.

The incremental kinetic energy is calculated so the user can examine the rate of change between timesteps:

$$U_{k_{inc}} = \frac{U_{k_n} - U_{k_{n-1}}}{\Delta t} \tag{3.11}$$

where U_{k_n} = kinetic energy at timestep ***n***;

$U_{k_{n-1}}$ = kinetic energy at timestep ***n-1***; and

Δt = timestep.

Kinetic energy is also calculated for rigid blocks in a similar fashion, based on the block mass and velocity.

3.3.4 Damped Energy (W_k)

The mass-damping work is the summation of all energy absorbed by either local damping or adaptive global damping (auto damping), and is intended for use primarily with static analysis. For dynamic analyses, the work done on viscous boundaries will be much larger than the damped energy, and will largely control the calculated value of the total released energy.

The damped energy can most easily be seen by examining a simplified version of the equation of motion,

$$\frac{\partial \dot{u}}{\partial t} = \frac{\sum F}{m} - \alpha \, \dot{u} \tag{3.12}$$

where $\dot{u}$ = velocity of a gridpoint of mass, m

$\sum F$ = the force sum at the gridpoint; and

α = damping coefficient, = $2\pi f \gamma$, where γ = fraction of critical damping; and f = natural frequency of the system (cps); the circular frequency in radians per second is $\omega = 2\pi f$

The damping force is given by

$$F_d = m\ \alpha\ \dot{u} \tag{3.13}$$

and the rate of damped energy change at a gridpoint is

$$\dot{W}_d = F_d\ \dot{u} = m\ \alpha\ \dot{u}^2 \tag{3.14}$$

The damped energy over a timestep at a gridpoint, ***j***, is

$$W_{dj} = \int \alpha\ m\ \dot{u}^2\ dt = 2\alpha\ \Delta t\ U_k \tag{3.15}$$

where W_{dj} is the energy damped at gridpoint ***j***, and U_k is the kinetic energy of the gridpoint.

Therefore, the damped and kinetic energy components are related by the damping coefficient. The total mass damping work is the sum of all gridpoints and timesteps:

$$W_d = \sum_{i=1}^{nt} \sum_{j=1}^{ngp} W_{dj} \tag{3.16}$$

where W_d = total energy damped;

nt = number of timesteps; and

ngp = number of gridpoints.

The incremental damped energy is also calculated as

$$W_{d_{inc}} = 2\alpha\ U_k \tag{3.17}$$

3.3.5 Strain Energy Stored in the Rock Mass (U_c)

The total strain energy stored in the rock mass is composed of two parts: the energy stored in the blocks, and that stored in the joints. Each is calculated, and the total stored energy, U_c, is determined as the sum of these two components.

3.3.5.1 Block-Stored Strain Energy (U_{cb})

The strain energy in the blocks is determined for all finite difference zones during each timestep. The total stored strain energy is calculated by summing the values for all blocks. The incremental strain energy in each zone for a timestep is given by

$$\Delta U_{c_z} = \frac{A}{2} \left[\, (\sigma_{11} + \sigma'_{11}) \, e_{11} + 2 \, (\sigma_{12} + \sigma'_{12}) \, e_{12} + (\sigma_{22} + \sigma'_{22}) \, e_{22} \, \right] \tag{3.18}$$

where $\sigma_{11}, \sigma_{22}, \sigma_{12}$ = current zone stresses;

$\sigma'_{11}, \sigma'_{22}, \sigma'_{12}$ = zone stresses from the previous timestep;

e_{11}, e_{22}, e_{12} = incremental strains over the current timestep; and

A = area of zone.

The strain energy in a block is the sum of all zones within the block,

$$U_{c_b} = \sum U_{c_z} \tag{3.19}$$

The total strain energy in a given timestep is the sum for all blocks. A running total is kept for all timesteps, so that the value of U_{c_b} will approach a constant value with time as the system approaches equilibrium.

The incremental block strain energy is also calculated by

$$U_{c_{b_{inc}}} = \frac{U_{c_b}}{\Delta t} \tag{3.20}$$

An exception to the above calculation method is used in the case of initial equilibrium of the model prior to any excavation. Normally, the in-situ stresses are set up in the model by "freezing" the stresses in each zone using the **insitu stress** command. By also applying boundary stresses (which are in equilibrium with the internal stresses) at the same time, the model will be at initial stress equilibrium and will not require any timestepping. However, this method of consolidating the body under initial stresses produces no strain in the body, and no apparent strain energy. Therefore, the alternate form of the strain-energy density equation is used to define the initial strain energy in the system:

$$U_{c_z} = \frac{A}{2E} \left[\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - 2\nu \, (\sigma_{11}\,\sigma_{22} + \sigma_{11}\,\sigma_{33} + \sigma_{22}\,\sigma_{33}) + 2\,(1+\nu)\,\sigma_{12}^2 \right] \qquad (3.21)$$

3.3.5.2 Joint Strain Energy (U_{c_j})

The strain energy stored in the joints is separated into four component parts for elastic strain in shear (U_{js}), compression (U_{jc}), tension (U_{jt}) and energy dissipated in slip (U_{jf}). The calculations of these components are performed differently for the two built-in joint constitutive models (the Coulomb slip model and the continuously yielding model). In the Coulomb model, the joint normal and shear stiffnesses are linear, whereas the continuously yielding model allows nonlinear stiffness. The energy is determined for each contact along all joints in the model, although, at present, it is not possible to separate energy by joint.

Coulomb Slip Model, Linear Stiffness

If $f_n < 0$,

$$U_{jt} = -\frac{1}{2}\,(f_n + f_n')\,u_n \qquad (3.22)$$

If $f_n \geq 0$,

$$U_{jc} = -\frac{1}{2}\,(f_n + f_n')\,u_n \qquad (3.23)$$

If $f_s < f_{s\,\max}$,

$$U_{js} = -\frac{1}{2}(f_s + f_s')\,u_s \qquad (3.24)$$

If $f_s \geq f_{s\,\max}$,

$$U_{jf} = \frac{1}{2}(f_s + f_s')\,u_s \qquad (3.25)$$

where f_n, f_s = current normal shear force at a contact, compression positive;

f_n', f_s' = previous normal shear forces at a contact;

u_n, u_s = incremental normal and shear displacements at the contact over the current timestep; and

$f_{s\,\max}$ = shear stress at which the Coulomb slip condition is met ($f_{s\,\max} < f_n \tan\phi + C$).

For the case in which $f_s \geq f_{s\,\max}$ and slip occurs, energy is dissipated in the Coulomb model by friction (heat). (See Section 3.3.5.4.)

Continuously Yielding Model – Nonlinear Stiffness Allowed

$$U_{jt} = 0 \;\; \text{no tension in CY model} \tag{3.26}$$

$$U_{jc} = -\frac{1}{2}\,(f_n + f_n')\,u_n \tag{3.27}$$

$$U_{js} = -\frac{F}{2}\,(f_s + f_s')\,u_s \tag{3.28}$$

$$U_{jf} = \frac{1-F}{2}\,(f_s + f_s')\,u_s \tag{3.29}$$

where f_n, f_s = normal shear forces at a contact;

f_n', f_s' = previous contact stresses;

u_n, u_s = incremental, normal shear displacements over the current timestep; and

F = the yielding factor for the continuously yielding model as described in Section 2 in **Constitutive Models**.

The total energy absorbed and dissipated by the joints, U_{cj}, is given by the sum of all components:

$$U_{cj} = U_{js} + U_{jc} + U_{jt} \tag{3.30}$$

3.3.5.3 Strain Energy Content of Excavated Material (U_m)

When rock is excavated, the strain energy that was stored in the excavated volume is released. *UDEC* allows excavation of the rock blocks that form the opening through use of the **delete** command or assignment of the null constitutive model (via **zone model null** or **change cons 0**). The null model does not delete the blocks, but forces in null blocks are prevented from being passed to gridpoints of adjoining blocks. The null zones can collapse due to deformation of the opening, and can later be changed to a backfill material. When a block is deleted or given a null constitutive model, the energy sums are updated.

The total strain energy in the excavated material consists of the strain energy in the blocks and the joints. The strain energy in the blocks is calculated in the same manner as the strain energy described previously:

$$U_{m_b} = \sum_{i=1}^{nb} \sum_{j=1}^{nz} \frac{A}{2E} \left[\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - 2\nu \left(\sigma_{11}\,\sigma_{22} + \sigma_{11}\,\sigma_{33} + \sigma_{22}\,\sigma_{33} \right) + 2(1+\nu)\sigma_{12}^2 \right]_j \tag{3.31}$$

where U_{mb} = block strain energy;

A = area of zone;

E = Young's modulus of the rock mass;

ν = Poisson's ratio;

$\sigma_{11}, \sigma_{22}, \sigma_{33}$ = principal stresses in zone centroid;

nb = number of blocks in excavated material; and

nz = number of zones in the block.

The total strain energy in the joints bounding the excavated block(s) (U_{cj}) is given as follows.

Coulomb Joints (Constant Stiffness)

$$U_{cj} = \sum_{i=1}^{nc} + \frac{1}{2} \left[\frac{f_n^2}{k_n} + \frac{f_s^2}{k_s} \right]_i \tag{3.32}$$

where U_{cj} = strain energy stored in the joints;

f_n, f_s = normal shear force in joints;

k_n, k_s = normal shear stiffness of joints; and

nc = number of contacts.

Continuously Yielding Joints (Nonlinear Normal Stiffness)

$$U_{cj} = \sum_{i=1}^{nc} \frac{1}{2} \left[\frac{u_n^{1+\frac{1}{1-e_n}} \, [a_n \, (1-e_n)]^{\frac{1}{1-e_n}}}{1+\frac{1}{1-e_n}} + \frac{f_s^2}{k_s} \right]_i \tag{3.33}$$

where e_n = normal stiffness exponent;

u_n = normal displacement (closure) of joint surfaces;

a_n = initial normal stiffness of joint;

f_s = shear force at contact;

k_s = shear stiffness of contact; and

nc = number of contacts.

When a block is excavated, its energy is removed from the total strain energy, U_c, and added to the total for the excavated material, U_{mb}. The initial or in-situ strain energy state for the rock mass is determined by using the standard strain-energy density function, where the principal stresses are equal to the in-situ stresses. This is added to the boundary loading work (W) for the initial equilibrium, pre-mining condition. The final values for stored strain energies are determined:

$$U_c = U_c' - U_{mj} - U_{mb} \tag{3.34}$$

$$U_m = U_m' + U_{mj} + U_{mb} \tag{3.35}$$

$$U_{cb} = U_{cb} - U_{mb} \tag{3.36}$$

$$U_{cj} = U_{cj} - U_{mj} \tag{3.37}$$

where a “ ′ ” refers to the values from previous excavation steps.

3.3.5.4 Friction Work Done on Joints (W_j)

Energy is dissipated through frictional heating of joints. This work done is exchanged from the elastic strain energy, and is irrecoverable. *UDEC* keeps track of the frictional energy separately from the elastic (stored) joint energy terms (U_{jt}, U_{jc} and U_{js}). The friction loss is calculated for linear and nonlinear normal stiffness as follows.

Coulomb Joints, Linear Normal Stiffness

If $f_s \geq f_{s\,\max}$,

$$U_{jf} = \sum_{i=1}^{nc} \frac{1}{2} \, (f_s + f_s') \, u_s \tag{3.38}$$

where U_{jf} = frictional energy at the contact during a timestep;

f_s = current shear force at a contact;

f_s' = previous shear force at a contact;

u_s = increment in shear displacement; and

nc = number of contacts.

Continuously Yielding Joint (Nonlinear Normal Stiffness)

$$U_{jf} = \sum_{i=1}^{nc} \frac{1-F}{2} \, (f_s + f_s') \, u_s \tag{3.39}$$

where F = the yield factor for the continuously yielding joint, and nc = number of contacts.

The total dissipated energy is kept by summing over all the timesteps during an excavation step,

$$W_j = \sum_{i=1}^{nt} U_{jf} \tag{3.40}$$

where W_j = total dissipated friction energy, and nt = number of timesteps.

3.3.6 Viscous Boundary Work (W_v)

Viscous boundaries are used to dampen reflections of incident stress waves. The energy damped from these stress waves is calculated from the boundary forces and deflections at the boundary gridpoints:

$$W_{gj} = f_x \, u_x + f_y \, u_y \tag{3.41}$$

where W_{gj} = boundary work at a gridpoint, $\boldsymbol{j}$;

f_x, f_y = boundary forces; and

u_x, u_y = boundary displacements.

The viscous energy for a timestep is given by

$$W_v = \sum_{j=1}^{nbp} W_{gj} \tag{3.42}$$

where nbp = number of boundary gridpoints.

The total viscous work is summed for all timesteps. The incremental viscous boundary work is calculated by

$$\Delta W_v = \frac{W_v}{\Delta t} \tag{3.43}$$

where Δt = the timestep.

3.3.7 Energy Dissipation in Blocks through Plastic Work (W_p)

Several plasticity models that can describe the deformability of the blocks are available in *UDEC*. Energy is dissipated through plastic work as the zones undergo irreversible deformation. The strain in any zone can be divided into an elastic and a plastic part. The elastic strain can be determined, followed by the elastic strain energy as determined previously. The plastic work is found by taking the difference between the total strain energy and the elastic energy component.

The elastic strain energy is given by

$$W_e = \frac{A}{2E} \left[\sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 - 2\nu \, (\sigma_{11} \, \sigma_{22} + \sigma_{11} \, \sigma_{33} + \sigma_{22} \, \sigma_{33}) + 2(1+\nu)\sigma_{12}^2 \right] \tag{3.44}$$

The *change* in elastic strain energy between excavation steps is given by

$$\Delta W_e = W_e - W_e' \tag{3.45}$$

where W_e is the current elastic strain energy, and W_e' is the previous elastic strain energy.

The *total* energy change can be found from the total strain and stress,

$$\Delta W_T = \frac{A}{2}\left[\,(\sigma_{11} + \sigma_{11}')\, e_{11} + 2\,(\sigma_{12} + \sigma_{12}')\, e_{12} + (\sigma_{22} + \sigma_{22}')\, e_{22}\right] \tag{3.46}$$

The total plastic work dissipated during an excavation step is the difference between the total and elastic energy change at any timestep,

$$\Delta W_p = \Delta W_T - \Delta W_e \tag{3.47}$$

and the total dissipated is simply the sum of ΔW_p for all blocks at each timestep.

3.3.7.1 Energy Dissipated in Backfill Compression

UDEC keeps track of those blocks that have been excavated using the null model method and replaced by backfill. In this case, elastic strain energy may be stored in the fill, and energy may be dissipated via plastic deformation. This would normally be the case for sandfills. Energy values are calculated as either stored or dissipated for each zone (as previously described), and are added to the plastic work term.

3.3.7.2 Volume of Excavated Material (V_m)

When a block is deleted or assigned a null constitutive model type, the volume is added to the value V_m. The area is calculated by using the mass and density of the zones that compose it:

$$A_z = \text{mass/density}$$

where A_z = area of a zone.

The volume of the deleted blocks is then equal to

$$V_m = \sum A_z \tag{3.48}$$

for all zones in the excavated blocks.

3.4 Method of Operation in *UDEC*

The energy calculations in *UDEC* are initiated using **config energy** and the **block mechanical energy on** command. From this point on, the energies described in the previous sections are calculated in an incremental fashion at each timestep from the stress, force, displacement and strain changes. All energy values are summed from this point, with the exception of the kinetic energy, U_k, which is kept as an incremental value. Therefore, the magnitude of the energies upon printout will be the sum for the problem since timestepping began, and will include that computed for all excavation steps. Several points are noted:

1. The U_m component of energy is calculated immediately as blocks are deleted.
2. The mass-scaling option in *UDEC* must be disabled (by specifying **block mechanical mass-scaling off**), as it artificially adjusts masses of gridpoints to speed convergence.
3. If boundary element coupling is used to represent the outer boundary, stress units must be in terms of MPa.

Two additional commands have been added to *UDEC* for plotting and printing these energy components. The command **block mechanical history energy-sum** will keep time histories of all energy components, and **block mechanical list energy** will provide a summary listing of all energy components in table form.

Section 3.5 presents an example that illustrates the energy monitoring calculations in *UDEC*.

3.5 Energy Calculations: Excavation of a Circular Hole in an Infinite Elastic Medium

Salamon (1984) solved the problem of the stored and released energy in creation of a circular tunnel in an infinite, elastic rock mass subjected to hydrostatic stresses. Here, *UDEC* is used to generate energy components for this example, and the results are compared to the analytical solution. Salamon assumes an infinite rock mass, such that the tractions and displacements induced by the excavation of the hole become vanishingly small as the distance from the opening becomes large. However, the following must be considered in this problem.

(1) As the boundary approaches infinity, the induced tractions and displacements approach zero, but the area of the surface over which these tractions act approaches infinity.

(2) For a finite boundary, the tractions and displacements are not zero, and their dot product (work) is a scalar. Therefore, the *work* done by external forces cannot be canceled like the tractions.

Since the *UDEC* model is of finite size, the induced tractions and displacements are not zero, and the outer boundary of the model must be taken into account in determining stored strain energy and boundary work components. The following section reviews the Salamon solution and the changes necessary for determination of these two components.

3.5.1 Derivation of Analytical Solution to Cylindrical Tunnel in an Infinite Medium

The derivation of the energy equations is given for the analytical solution to a cylindrical tunnel in an infinite medium. Consider the 2D section shown in Figure 3.1, which has a Stage I excavated radius of a and Stage II radius of c. The boundary is located a distance R from the center of the tunnel. The tunnel is assumed to be sufficiently long that there are no end effects.

Solving the problem requires a few definitions. First, the volume of rock to be mined is $V_m = \pi(c^2 - a^2)$ per unit length of tunnel. The stress distribution at any point around the tunnel is given by the radial stress, $\sigma_r^{(p)}$, and tangential stress, $\sigma_t^{(p)}$, as follows (Jaeger and Cook 1979).

$$\begin{aligned} \sigma_r^{(p)} &= p\left(1 - \frac{a^2}{r^2}\right) \\ \sigma_t^{(p)} &= p\left(1 + \frac{a^2}{r^2}\right) \end{aligned} \tag{3.49}$$

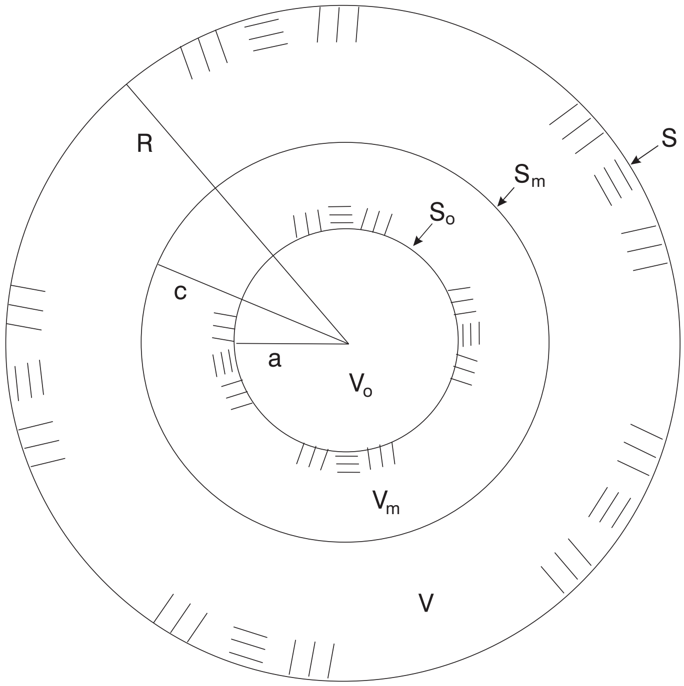

Figure 3.1 Cross section through an infinite length tunnel

in which r is the distance from the tunnel center to the point of interest;

p is the virgin hydrostatic compression stress; and

a is the Stage I excavated radius.

Assuming plane strain conditions, the strains are related to stresses by

$$\epsilon_r^{(p)} = \frac{p}{2G}\left[(1-2\nu) - \frac{a^2}{r^2}\right] \tag{3.50}$$

$$\epsilon_T^{(p)} = \frac{p}{2G}\left[(1-2\nu) + \frac{a^2}{r^2}\right]$$

The induced stresses that result from going from Stage I to Stage II equilibrium are given by subtracting stresses in Eq. (3.49) at excavated radius c from those at a. These stresses are given by

$$\sigma_r^{(i)} = -p\left(\frac{c^2-a^2}{r^2}\right) \tag{3.51}$$

$$\sigma_t^{(i)} = \ p\left(\frac{c^2-a^2}{r^2}\right)$$

The induced strains are determined in the same manner, and are given by

$$\epsilon_r^{(i)} = -\frac{p}{2G}\left(\frac{c^2 - a^2}{r^2}\right) \tag{3.52}$$

$$\epsilon_t^{(i)} = \frac{p}{2G}\left(\frac{c^2 - a^2}{r^2}\right)$$

Displacements at any point around the tunnel during Stage I are given by Jaeger and Cook (1979):

$$u_r^{(p)} = \frac{p}{2G}\left(\frac{a^2}{R}\right) \tag{3.53}$$

$$u_t^{(p)} = 0$$

Similar expressions are used for Stage II displacements, except that a^2 is replaced by c^2.

Given the above definitions, it is now possible to solve for the energy terms. The equation relating the change in potential energy to the work done by the body in going from Stage I to Stage II is given by (after Salamon 1984, Eq. (18a))

$$W + U_m = U_c + W_r \tag{3.54}$$

where W = work done by external and body forces when acting through the induced displacements;

U_m= stored strain energy in the mined rock volume V_m at Stage I;

U_c = change in stored strain energy in the unmined rock volume V;

W_r= released energy, = $U_m + W_k$; and

W_k= kinetic energy dissipated by damping in the unmined rock and supports.

The stored strain energy in the volume of rock mined, U_m, at Stage I can be defined in terms of a strain energy density function, ϕ. The strain energy density function is defined as

$$\phi = \frac{1}{2}\,\sigma_{ij}\,\epsilon_{ij} \tag{3.55}$$

which, according to Jaeger and Cook (1979), can be expressed in terms of the stress tensor as

$$\phi = \frac{1}{2E}\left[\,\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu\,(\sigma_y\sigma_z + \sigma_z\sigma_x + \sigma_x\sigma_y) + 2(1+\nu)(\tau_{xy}^2 + \tau_{zx}^2 + \tau_{zy}^2)\,\right] \tag{3.56}$$

For the case of two-dimensional plane strain, two of the three shear stress components are zero ($\tau_{xz} = \tau_{yz} = 0$), and the strain energy density function in Eq. (3.56) reduces to

$$\phi = \frac{1}{2E}\left[\,\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu\,(\sigma_y\sigma_z + \sigma_z\sigma_x + \sigma_x\sigma_y) + 2\,(1+\nu)(\tau_{xy}^2)\,\right] \tag{3.57}$$

The stored strain energy in the mined volume of rock, U_m, at Stage I is the integral of the energy density function over the entire volume, V_m; that is (Salamon 1984, Eq. (14)),

$$U_m = \int_{V_m} \phi\, dV \tag{3.58}$$

By substituting Eqs. (3.49) and (3.50) into Eq. (3.58), the stored strain energy in the rock to be mined is

$$\begin{aligned} U_m &= \int_{V_m} \phi^I\, dV = \int_0^{2\pi}\int_0^c r\left[\,\sigma_r^{(p)}\,\epsilon_r^{(p)} + \sigma_t^{(p)}\,\epsilon_t^{(p)}\,\right] dr\,d\theta \\ &= \frac{p^2}{2G}\left[\,\frac{r^2}{2}\,(1-2\nu)\,\theta - \frac{a^4}{2r^2}\,\theta\,\right]\Big|_a^c\,\Big|_0^{2\pi} \\ &= \frac{p^2}{2G}\left[\,(1-2\nu) + \frac{a^2}{c^2}\,\right] V_m \end{aligned} \tag{3.59}$$

This term is the same as given by Salamon (1984), and is independent of the boundary radius.

The work done by external and body forces, W, is often referred to as the "gravitational" or "potential" energy change. The change in potential energy is the sum of external work, W_{ext}, plus the work done by body forces, W_{body}. The work done by external forces, W_{ext}, can be expressed in terms of surface tractions, T_i. The work done by the body forces, W_{body}, is expressed in terms of force per unit volume, X_i. This work is given as (Salamon 1984, Eq. (7))

$$\begin{aligned} W &= W_{\text{ext}} + W_{\text{body}} \\ &= -\int_S \left(T_i^{(p)} + \frac{1}{2} T_i^{(i)} \right) u_i^{(i)}\, dS + \int_V X_i\, u_i^{(i)}\, dV \end{aligned} \tag{3.60}$$

in which $T_i^{(i)}$ $= T_i - T_i^{(p)}$;

$u_i^{(i)}$ $= u_i - u_i^{(p)}$;

$T_i^{(p)}$ = boundary tractions at Stage I;

T_i = boundary tractions at Stage II;

$T_i^{(i)}$ = induced tractions from Stage I to II;

$u_i^{(p)}$ = boundary displacements at Stage I;

u_i = boundary displacements at Stage II;

$u_i^{(i)}$ = induced displacements from Stage I to II;

X_i = body forces per unit volume in volume V at Stage II; and

$X_i^{(p)}$ = body forces per unit volume in volume V at Stage I.

Given the identities in Eq. (3.60) for T_i and u_i, the work done by external forces can be rewritten as

$$W_{\text{ext}} = -\frac{1}{2} \int_S \left(T_i + T_i^{(p)} \right) \left(u_i - u_i^{(p)} \right) dS \tag{3.61}$$

The change in potential energy, W, is a function only of the work done by the external forces, W_{ext}, as it has been assumed that there are no body forces; hence, W_{body} is zero. Substituting relations for tractions, as given by stresses in Eq. (3.49), and displacements in Eq. (3.53), the change in potential energy is

$$
\begin{aligned}
W = W_{\text{ext}} &= \int_S \left(T_i^{(p)} + \frac{1}{2}\, T_i^{(i)} \right) u_i^{(i)}\, dS \\
&= \int_0^{2\pi} \left[p \left(1 - \frac{a^2}{R^2} \right) + \frac{1}{2}\,(-p) \left(\frac{c^2 - a^2}{R^2} \right) \right] \frac{p}{2G} \left(\frac{c^2 - a^2}{R} \right) R\, d\theta \\
&= \frac{p^2}{2G} \left(2 - \frac{c^2 + a^2}{R^2} \right) V_m
\end{aligned}
\tag{3.62}
$$

This equation is different than Salamon's, but it is derived directly from the surface integral rather than by making substitutions relying on the assumption that $U^{(ip)} = U_m$.

The change in stored strain energy in the unmined rock, U_c, at Stage II is the difference in integrated energy density over the volume, V, at the equilibrium states of Stage I and Stage II. It is written as

$$
U_c = \int_V \left[\phi^{II} - \phi^{I} \right] dV \tag{3.63}
$$

The change in stored strain energy in the unmined rock, U_c, is obtained from Eq. (3.63). The stored strain energy at Stage II is given by

$$
\begin{aligned}
U = \int_V \phi^{II}\, dV &= \int_0^{2\pi} \int_c^R r \left[\frac{1}{2} \left(\sigma_r^p\, \epsilon_r^p + \sigma_t^p\, \epsilon_t^p \right) \right] dr\, d\theta \\
&= \frac{p^2}{2G} \left[\frac{r^2}{2}\,(1 - 2\nu)\,\theta - \frac{c^4}{2r^2}\,\theta \right] \Big|_c^R \Big|_0^{2\pi} \\
&= \frac{p^2}{2G} \left[(1 - 2\nu) + \frac{c^2}{R^2} \right] (R^2 - c^2)\,\pi
\end{aligned}
\tag{3.64}
$$

and the stored strain energy at Stage I is given by

$$
\begin{aligned}
U^{(pp)} = \int_V \phi^{I}\, dV &= \int_0^{2\pi} \int_c^R r \left[\frac{1}{2} \left(\sigma_r^p\, \epsilon_r^p + \sigma_t^p\, \epsilon_t^p \right) \right] dr\, d\theta \\
&= \frac{p^2}{2G} \left[\frac{r^2}{2}\,(1 - 2\nu)\,\theta - \frac{a^4}{2r^2}\,\theta \right] \Big|_c^R \Big|_0^{2\pi} \\
&= \frac{p^2}{2G}\,(R^2 - c^2)\,\pi \left[(1 + 2\nu) + \frac{a^4}{c^2\, R^2} \right]
\end{aligned}
\tag{3.65}
$$

Note that Salamon does not work out U or $U^{(pp)}$, but the equation for $U^{(pp)}$ is the same as Salamon's (1984) Eq. (II.5). Subtracting Eq. (3.65) from Eq. (3.64) gives

$$U_c = \frac{p^2}{2G}\,(c^2 + a^2)\left(\frac{1}{c^2} - \frac{1}{R^2}\right) V_m \tag{3.66}$$

This expression does not agree with Salamon's (1984) Eq. (II.12). To check this result, U_c can be calculated as given by Salamon (1984):

$$U_c = U^{(ii)} + 2U^{(pi)} \tag{3.67}$$

The induced stored strain energy is given by

$$\begin{aligned} U^{(ii)} &= \int_V \phi^{(ii)}\,dV = \int_0^{2\pi}\int_c^R r\left[\frac{1}{2}\left(\sigma_r^i\,\epsilon_r^i + \sigma_t^i\,\epsilon_t^i\right)\right] dr\,d\theta \\ &= \frac{p^2}{2G}\,(c^2 - a^2)\left(\frac{1}{c^2} - \frac{1}{R^2}\right) V_m \end{aligned} \tag{3.68}$$

The stored strain energy induced by Stage I forces on the displacements that occur in Stage II is given by

$$\begin{aligned} U^{(pi)} &= \int_V \phi^{(pi)}\,dV = \int_0^{2\pi}\int_c^R r\left[\frac{1}{2}\left(\sigma_r^p\,\epsilon_r^i + \sigma_t^p\,\epsilon_t^i\right)\right] dr\,d\theta \\ &= \frac{p^2}{2G}\,a^2\left(\frac{1}{c^2} - \frac{1}{R^2}\right) V_m \end{aligned} \tag{3.69}$$

U_c is obtained by substituting Eqs. (3.69) and (3.70) into Eq. (3.68).

The released energy, W_r, as given by Eq. (3.56), gives

$$\begin{aligned} W_r &= W + U_m - U_c \\ &= \frac{p^2}{G}\,(1 - \nu)\,V_m \end{aligned} \tag{3.70}$$

The kinetic energy, W_k, will be

$$W_k = W_r - U_m = \frac{p^2}{2G}\left(1 - \frac{a^2}{c^2}\right) V_m \tag{3.71}$$

3.5.2 UDEC Energy Calculation

The analytical solution described in Section 3.4 is compared to the *UDEC* model results for the case of the initial excavation (Stage I) and enlargement (Stage II) of a circular tunnel. The radius of the initial excavation is 1 m, and the enlargement produces a tunnel of 2 m radius. The outer radius of the *UDEC* model is 10 m. A hydrostatic compressive stress of 100 MPa exists prior to excavation. The elastic material has a shear modulus of 29.17 GPa and Poisson's ratio of 0.2.

Figure 3.2 shows the initial *UDEC* block geometry, consisting of a number of concentric circular blocks, and the zoning within the blocks. The *UDEC* data file to calculate the two excavation stages and monitor energy components is listed in Example 3.1.

Note that adaptive global damping (**block mechanical damping global**) is used for this calculation. As discussed in Note 12 in Section 3.9 in the **User's Guide**, this damping is more computationally efficient than local damping for an elastic analysis. Similar results for the energy components will also be calculated if local damping is used.

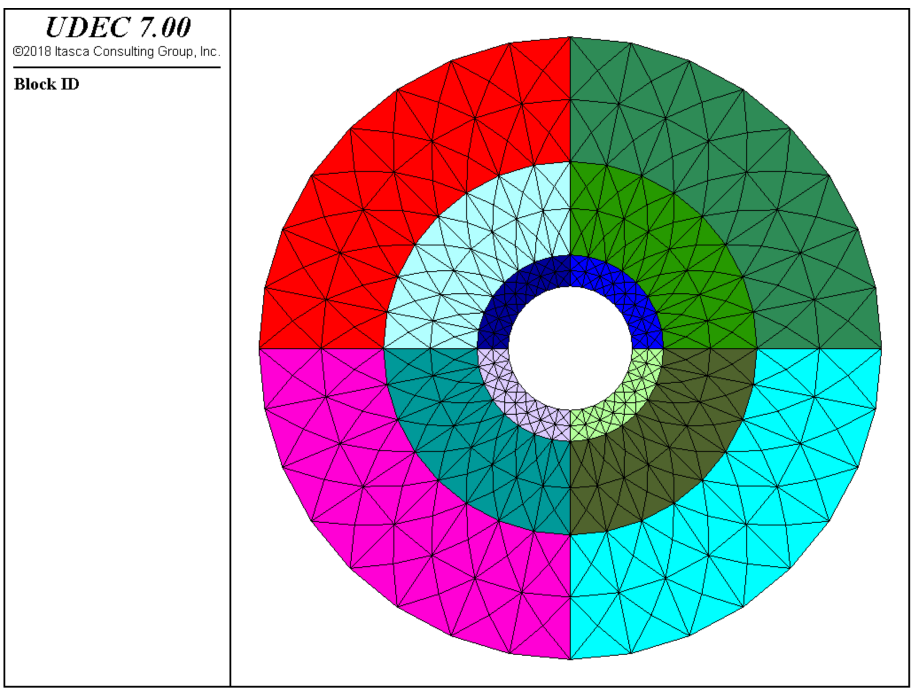

Figure 3.2 UDEC initial geometry

Example 3.1 Energy calculations for excavation of a circular hole

```
model new
block config energy
block mechanical energy off
model title 'ENERGY CALCULATION FOR CIRCULAR TUNNEL'
block tolerance corner-round-length 0.01
;use auto damping and turn off mass scaling
block mechanical damping global
block mechanical mass-scale off
;set geometry (before excavation)
block create circle 0 0 10 32
block cut crack -10 0 10 0
block cut crack 0 -10 0 10
block cut tunnel 0 0 1 32
block cut tunnel 0 0 2 32
block cut tunnel 0 0 3 32
block cut tunnel 0 0 6 32
;create zoning (4 different sizes)
block zone gen  quad=0.4 range ann center 0 0 rad 1 2
block zone gen  quad=0.8 range ann center 0 0 rad 2 3
block zone gen  quad=1.6 range ann center 0 0 rad 3 6
block zone gen  quad=3 range ann center 0 0 rad 6 10
block zone gen edge .4
;set stresses
block edge apply stress -100 0 -100
block insitu  stress -100 0 -100 stress-ZZ -40
;material properties
block property material 1 density .002 bulk 38.9e3 shear 29.17e3
block domain property material 1 capillary-gamma 29.17e3
block contact property material 1 friction 40.0 cohesion 10e8 ...
  tension 10e7 stiffness-normal 6e8 stiffness-shear 6e8
;histories (displacements and stresses) at radii=5,10,20
hist interval =20
block gridpoint history disp-x  1 0
block gridpoint history disp-x 2 0
block zone history      stress-xx  1 0
block zone history      stress-xx  2 0
block zone history      stress-xx  3 0
block zone history      stress-yy  1 0
block zone history      stress-yy  2 0
block zone history      stress-yy  3 0
model display hist 1
block solve rat 1e-5
model save 'energy0.sav'
```

```
block mechanical energy on
block mechanical hist energy-sum
block boun-el gen
block boun-el mat 1
block boun-el fix 0 -100 -100 0
block boun-el stiff
; excavation step 1
block del range ann center 0,0 rad 0 1
block solve rat 1e-5
log on
block list energy
log off
model save 'energy1.sav'

; excavation step 2
block del range ann center 0,0 rad 1,2
block solve rat 1e-5
log on
block list energy
log off
model save 'energy2.sav'
```

Figure 3.3 gives an example of a history plot for the damped and kinetic energy components for the first excavation stage. In this figure, the kinetic energy term is not summed over time, but decays to zero as the model comes to equilibrium. At the same time, damped energy, which is summed, approaches a constant value.

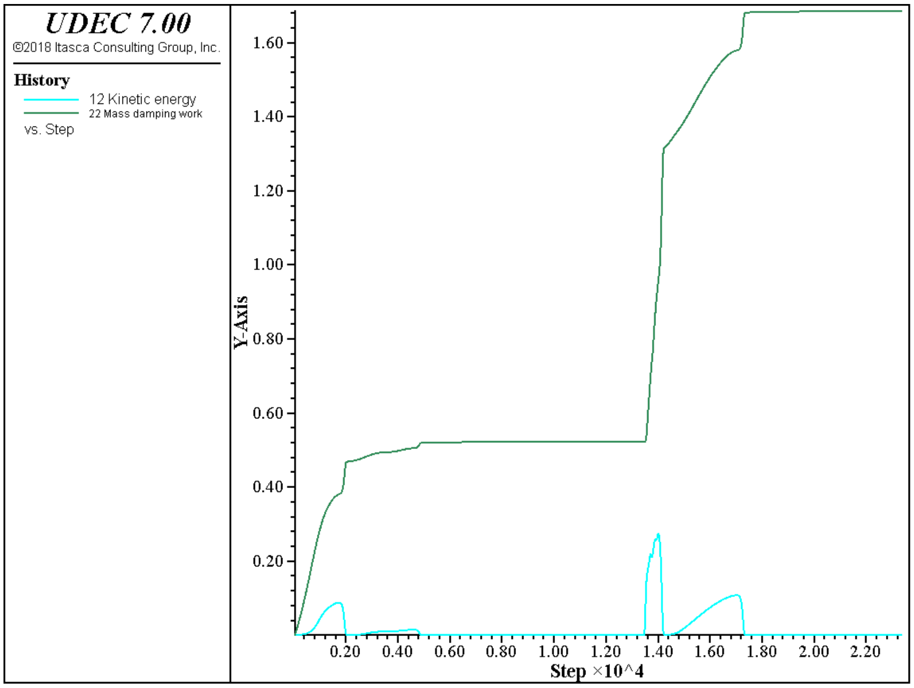

Figure 3.3 ***Plot of the history of the damped, W_k (hist 21), and incremental kinetic energy, U_k (hist 11), components. (The kinetic energy drops to zero as the model comes to equilibrium, whereas the damped (summed kinetic) energy approaches a constant value.)***

Table 3.1 shows a typical result of the **block mechanical list energy** command after the first stage of excavation. All of the current energy components and their rates of change are printed.

Table 3.1 Totals for energy stored and dissipated in system

current kinetic energy (Uk)	= 2.601E−06
total block strain energy (Ucb)	= 2.052E−01
total fill strain energy (Ucf)	= 0.000E+00
total joint strain energy (Ucj)	= −1.099E−04
total material strain energy (Uc=Ucb+Ucf+Ucj)	= 2.051E−01
total block energy excavated (Umb)	= 3.210E−01
total joint energy excavated (Umj)	= 8.699E−05
total strain energy excavated (Um=Umb+Umj)	= 3.211E−01
total block volume excavated (Vm)	= 3.121E+00
total change in potential energy (Ub)	= 0.000E+00
total mass damping work (Wk)	= 5.216E−01
total viscous boundary work (Wv)	= 0.000E+00
total friction work (Wj)	= 0.000E+00
total plastic strain work (Wp)	= 0.000E+00
total boundary loading work (W)	= 1.048E+00
total energy released (Wr=W-Uc-Ub-Wj-Wp)	= 8.431E−01
total energy released (Wr=Uk+Wk+Wv+Um)	= 8.427E−01
breakdown of energy stored in joints (Ucj)	
total energy stored in tension (Ujt)	= 0.000E+00
total energy stored in compression (Ujc)	= −1.099E−04
total energy stored in shear (Ujs)	= 6.461E−08

3.5.3 Comparison to Salamon Solution

Table 3.2 summarizes the results from the analytical solution and *UDEC*. The comparison is good, generally within 3%.

Table 3.2 Summary of results from the analytical solution and UDEC

ENERGY	*Excavation Stage I*			*Excavation Stage II*		
COMPONENT	**analytic**	**udec**	**error (%)**	**analytic**	**udec**	**error (%)**
U_c	0.533	0.532	0.375	1.939	1.871	3.507
U_m	0.323	0.321	0.623	1.373	1.354	1.384
V_m	3.141	3.121	0.637	9.425	9.369	0.594
W_k	0.538	0.522	2.974	1.212	1.165	3.878
W	1.071	1.048	2.148	3.150	3.046	3.302
W_{R1}	0.861	0.843	2.091	2.585	2.523	2.400
W_{R2}	0.861	0.843	2.091	2.585	2.518	2.592

NOTES

1. *UDEC* energy components at stage II are obtained by subtracting components at stage I from the total components reported with **block mechanical list energy** at stage II.
2. U_c in *UDEC* is different from that given by Salamon (1984). The relation is

$$U_c^{(UDEC)} = U_c^{(Salamon)} - U_m$$

3. $W_{R1} = W - U_c^{(UDEC)}$
4. $W_{R2} = W_k + U_m$

3.6 Reference

Jaeger, J. C., and N. G. W. Cook. ***Fundamentals of Rock Mechanics***, 3rd Ed. London: Chapman and Hall (1979).

Salamon, M. D. G. "Energy Considerations in Rock Mechanics: Fundamental Results," *J. S. Afr. Inst. Min. Metall.*, ***84***(8), 233-246 (1984).